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PROJECT RATRAN
RESEARCH ON THE ESTIMATION OF TRAJECTORIES FROM RADAR DATA

THIRD QUARTERLY REPORT

Prepared by
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S. Gottesman

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Philadelphia, Pennsylvania 19104

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FOREWORD

This report describes work done from 1 November 1971 through 31 January 1972 at the Moore School of Electrical Engineering, University of Pennsylvania, under contract number DAAB07-71-C-0212 with U. S. Army Electronics Command for research entitled "The Estimation of Trajectories from Radar Data". The cognizant technical personnel at USA ECOM are Dr. Leonard Hatkin, head of the Radar Technical Area and Mr. Donald Foiani, CSTA Laboratory, Evans Area, Fort Monmouth, N.J. 07703.

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1. INTRODUCTION

As indicated in the previous Quarterly Progress Reports [1,2], this work is concerned with the general signal processing problem of estimating characteristics and trajectory parameters of a body moving in the air from noisy radar observations.

The main thrust during this quarter has been to evaluate the simplified dynamics equations evolved, and to complete, refine, debug and conduct sample runs of computer programs to do the following tasks: a) obtain algorithms to perform optimal recursive smoothing and filtering from simulated noisy radar data; b) deterministic evaluation of expected filter-smoother algorithm performance; and c) determination of idealized or minimum variance sensitivity of trajectory parameter estimation accuracy to radar observation errors.

The evaluation of the Simplified Dynamics Equations is presented in Section 2 to follow. Revised values for universal drag and drift curves are presented, followed by a set of runs of a wide variety of projectile trajectories showing resultant backtracking errors due to the simplified dynamics.

The detailed formulation required to perform optimal recursive smoothing and filtering of noisy radar data was presented in Section 4 of ref. 2. The required mathematical derivation of the algorithm is included here in Section 3 below. Details of the simulation program, performance evaluation for tracking and parameter estimation of trajectories will be included in the final report.

In Section 4 is presented the derivation of a deterministic method for evaluation of expected filter-smoother algorithm performance. This technique can account for bias errors as well as random errors in radar observations, for both the optimal filter-smoother algorithm described in Section 3 and also arbitrary non-optimal recursive algorithms that may be employed.

The program for the idealized sensitivity study has been debugged and results are being prepared for the final report. These results will be useful in assessing the efficiency of the recursive algorithms developed as well as providing tradeoffs as to significance of different types of errors.

2.0 EVALUATION OF SIMPLIFIED DYNAMICS EQUATIONS

The development of simplified dynamics equations, incorporating approximations to drag and drift accelerations, was presented in ref. 2, Section 2. The work presented here consists of a refined set of numerical constants needed for the approximations used plus an extensive set of evaluation results for the four different projectiles considered.

The refined set of approximation constants is presented in Table 1. Three sets of polynomial coefficients are given defining "universal" curves for drag, drift, and spin.

Evaluations of the Simplified Dynamics approximations are presented in Tables 2a, b, c, and d for the 105mm, 155mm, 175mm, and 8-inch projectiles, respectively. In each case results are shown for a variety of charges and quadrant elevation angles. Computations of backtracking accuracy are performed for reversal times given in multiples of ten seconds.

The principal factors affecting the errors are (i) time of reversal, corresponding in the use of the system to the time at which the state vector estimate is made; (ii) quadrant elevation, and (iii) projectile type.

Item (ii) is a geometric sensitivity factor. For small quadrant elevation angle QE, the backtracking trajectory is nearly parallel to the ground as it nears the launch point. A small vertical error is magnified by the cosecant of the angle QE. The values of the error in X, in Table 2, multiplied by sin(QE) give values that cluster together much better than the values of the error as given in the Table. This is illustrated in Table 3, showing the ratio of ℓ_T, the normal distance of trajectory computed by the simplified dynamics from the true trajectory at the launch point, to ℓ_X, the longitudinal error in launch point due to the simplified dynamics approximations.

At 10 second reversal time, the largest value of (error in X)(sin QE) is 4.7 meters, for the 155mm shell at QE = 216 mils. At 20 second reversal time the largest value is 25.8 meters, for the same trajectory. If the 155mm shells are excluded, the largest error products are 2.5 and 15.6 meters respectively at 10 and 20 second reversal time, both for 105mm shells.

The sensitivity to projectile type suggests that the aerodynamic drag function fit could be tailored better to apportion the errors over the different projectiles, so that the 155mm and perhaps the 105mm errors would be decreased at the expense of the 175mm and 8-inch errors. For the latter two projectiles, the largest values of (error in X)(sin QE) were 2.1 meters at 10 sec. reversal time and 12.5 meters at 20 sec. reversal time, both for the 175mm shell at QE = 244.

For all the cases studied, the importance of backtracking from a smoothed state vector <u>as soon as possible</u> is apparent. The drag approximation leads to unacceptably large errors for reversal times much in excess of ten seconds. This points out the importance of the "smoothing" as opposed to "filtering" features of the recursive algorithm being developed.

UNIVERSAL DRAG CURVE

A(0)	A(1)	A(2)	A(3)	A(4)	UPPER MACH NO BREAK POINT
+27754[illegible]-0[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	+[illegible]
294074[illegible]-0[illegible]	-[illegible]8556[illegible]9-[illegible]2	35[illegible]1[illegible]9-02	-4078[illegible]18[illegible]-[illegible]2	[illegible]54[illegible]2-0[illegible]	9[illegible]0000 0[illegible]
-56[illegible]92[illegible]7[illegible] 0[illegible]	24140455 [illegible]1	-3[illegible]836[illegible]25 01	27445913 [illegible]	-73[illegible]0 0[illegible]	99[illegible] 0[illegible]
1649142 0[illegible]	-64722510 [illegible]1	95427249 01	-62486[illegible]3[illegible] 01	153[illegible]78[illegible]7 01	1[illegible]0000 01
-36991679-0[illegible]	12251269-01	-14[illegible]8277-01	70647332-[illegible]2	-13523435-0[illegible]	15[illegible]00000 01
17693159-0[illegible]	-14842065-03	7[illegible]643028-04	-21621397-[illegible]4	[illegible]6798426-0[illegible]	50[illegible]00000 01

UNIVERSAL [illegible] CURVE

A(0)	A(1)	A(2)	A(3)	A(4)	UPPER MACH NO BREAK POINT
9417802[illegible]-0[illegible]	-20634522-[illegible]2	61890538-02	-32734184-02	-3[illegible]7952-02	84[illegible]0000 00
105465[illegible] 0[illegible]	-4[illegible]6644[illegible]9 02	82211535 02	-52499311 02	1781[illegible]323 0[illegible]	95[illegible]0000 00
2661537[illegible] 0[illegible]	-10493467 03	15[illegible]5378 03	-10156[illegible]64 03	249357[illegible]1 0[illegible]	1[illegible]00000 01
-48373562 0[illegible]	14761935 01	-16[illegible]6541 01	[illegible]2453192 00	-15[illegible]94547 0[illegible]	15[illegible]00000 01
15625[illegible]6-0[illegible]	-16050[illegible]4-01	17687456-01	-69302355-02	56158[illegible]2-0[illegible]	40[illegible]00000 01

UNIVERSAL SPI[illegible] CURVE

A(0)	A(1)	A(2)	A(3)	A(4)	UPPER MACH NO BREAK POINT
+70000[illegible]0-0[illegible]	-2[illegible]504608-02	-9[illegible]103102-03	+25784890-02	-11479416-0[illegible]	+9[illegible]0000+ 0
+67249370-0[illegible]	-24994776-02	+7[illegible]8136-03	-12021482-[illegible]3	+50635545-0[illegible]	+50000000+ 1

TABLE 1 Universal Curve Coefficients*

*Supersedes Table B.2 in Second Quarterly Report

INITIAL VELOCITY (M/S)	MACH NO. AT INITIAL VELOCITY	QUADRANT ELEVATION (MILS)	APPROXIMATE TIME AT WHICH MACH 1 IS PASSED (SEC)	TIME AT REVERSAL (SEC)	HEIGHT AT REVERSAL (M)	DRIFT AT REVERSAL (M)	SIMPLIFIED DYNAMICS INCLUDING DRAG AND DRIFT ERROR IN X (M)	ERROR IN Z (M)
465.0	1.3665	200	5.5	10	284.056	9.897	12.597	1.639
234.0	0.6876	300	—	10	174.253	8.759	5.928	0.291
465.0	1.3665	300	5.5	10 20	636.413 233.142	10.078 40.368	8.391 -45.680	1.506 0.374
234.0	0.6876	422	—	10	424.542	8.749	4.797	-0.003
465.0	1.3665	368	5.5	10 20	872.559 666.188	9.998 40.731	5.719 -21.988	1.411 0.518
465.0	1.3665	269	5.5	10 20	528.472 37.845	10.063 40.037	9.498 -61.419	1.543 0.320

Table 2a Backtracking Errors Due to Simplified Dynamics Approximations for 105mm Shells

INITIAL VELOCITY (M/S)	MACH NO. AT INITIAL VELOCITY	QUADRANT ELEVATION (MILS)	APPROXIMATE TIME AT WHICH MACH 1 IS PASSED (SEC)	TIME AT REVERSAL (SEC)	HEIGHT AT REVERSAL (M)	DRIFT AT REVERSAL (M)	SIMPLIFIED DYNAMICS INCLUDING DRAG AND DRIFT ERROR IN X (M)	SIMPLIFIED DYNAMICS INCLUDING DRAG AND DRIFT ERROR IN Z (M)
371.9	1.0929	272	2.0	10	405.045	11.752	8.342	-0.747
463.3	1.3615	391	6.8	10	1001.265	13.920	9.888	-0.628
				20	903.151	53.121	47.177	-2.641
463.3	1.3615	216	7.3	10	369.438	14.061	22.564	-0.585
563.9	1.6571	270	11.7	10	767.307	14.988	7.291	0.026
				20	455.920	56.560	96.604	-3.171
563.9	1.6571	410	11.3	10	1363.225	14.926	5.270	0.030
				20	1497.430	57.379	54.580	-3.227
				30	686.201	124.369	132.158	-6.610
563.9	1.6571	522	11.1	10	1820.796	14.519	4.156	0.025
				20	2314.331	56.743	39.054	-2.977
				30	1828.945	125.488	119.716	-4.064
				40	451.209	215.070	218.592	-6.134

Table 2b Backtracking Errors Due to Simplified Dynamics Approximations for 155 mm Shells

INITIAL VELOCITY (M/S)	MACH NO. AT INITIAL VELOCITY	QUADRANT ELEVATION (MILS)	APPROXIMATE TIME AT WHICH MACH 1 IS PASSED (SEC)	TIME AT REVERSAL (SEC)	HEIGHT AT REVERSAL (M)	DRIFT AT REVERSAL (M)	SIMPLIFIED DYNAMICS INCLUDING DRAG AND DRIFT ERROR IN X (M)	ERROR IN Z (M)
704.1	2.0691	244	26.0	10	980.521	13.915	8.838	-2.063
				20	864.696	50.300	52.473	-2.569
510.5	1.5002	155	14.8	10	225.524	10.477	13.152	0.043
914.4	2.6871	269	38.5	10	1563.121	17.644	3.030	-4.312
				20	1865.623	63.930	44.459	-11.937
				30	1238.479	129.930	144.119	-12.289
914.4	2.6871	291	39.5	10	1728.747	17.346	2.730	-4.252
				20	2164.605	63.248	40.455	-11.904
				30	1645.555	129.525	130.576	-12.888
				40	373.257	207.416	-352.868	-0.354
914.4	2.6871	640	—	10	4125.238	16.814	1.311	-3.777
				20	6498.496	63.822	16.855	-11.897
				30	7622.434	138.191	56.828	-18.442
				40	7702.371	237.702	118.832	-14.233
914.4	2.6871	772	—	10	4921.148	15.105	1.130	-3.417
				20	7988.375	58.446	13.124	-11.150
				30	9763.125	129.484	45.591	-18.182
				40	10453.473	228.798	100.474	-16.658
914.4	2.6871	500	57.7	10	3202.578	16.852	1.650	-4.079
				20	4797.785	63.383	22.513	-12.315
				30	5224.102	135.042	74.276	-17.548
				40	4687.930	227.100	159.200	-8.025
914.4	2.6871	422	47.5	10	2665.320	17.675	1.963	-4.158
				20	3829.218	65.339	26.934	-12.229
				30	3892.313	136.553	87.457	-16.201
				40	3060.573	224.640	19.872	-6.090
914.4	2.6871	828	—	10	5231.551	15.166	1.072	-3.262
				20	8571.871	58.620	11.989	-10.769
				30	10606.523	130.114	41.975	-17.818
				40	11545.785	230.874	95.446	-16.406

Table 2c Backtracking Errors Due to Simplified Dynamics Approximations for 175mm Shells

INITIAL VELOCITY (M/S)	MACH NO. AT INITIAL VELOCITY	QUADRANT ELEVATION (MILS)	APPROXIMATE TIME AT WHICH MACH 1 IS PASSED (SEC)	TIME AT REVERSAL (SEC)	HEIGHT AT REVERSAL (M)	DRIFT AT REVERSAL (M)	SIMPLIFIED DYNAMICS INCLUDING DRAG AND DRIFT ERROR IN X (M)	ERROR IN Z (M)
420.6	1.2360	400	5.7	10	946.841	10.193	0.317	-0.209
				20	836.312	38.420	8.738	1.065
594.4	1.7467	400	15.5	10	1463.583	11.843	2.344	-0.500
				20	1691.349	47.402	-14.130	-2.154
				30	958.470	102.905	66.072	1.482
304.8	0.8957	400	—	10	643.057	7.649	3.143	0.123
				20	296.649	33.295	15.569	0.338

Table 2d Backtracking Errors Due to Simplified Dynamics Approximations for 8-inch Shells

Quadrant Elevation mils	Sin(QE) (ℓ_T/ℓ_X)
200	.195
300	.290
400	.382
500	.471
600	.555
700	.635
800	.706

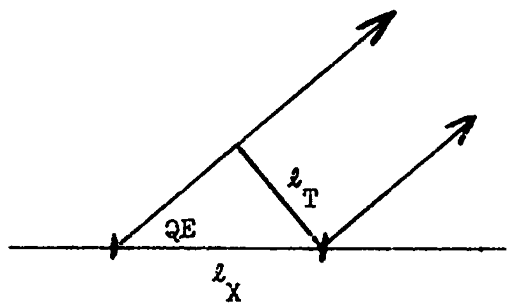

Table 3 Ratio of Trajectory Error ℓ_T to X-coordinate Error ℓ_X as a Function of Quadrant Elevation Angle

3.0 DEVELOPMENT OF ESTIMATION ALGORITHMS

3.1 Introduction. This section will be devoted to the development of the nonlinear filter-smoothing algorithm presented in the second quarterly report.* The derivation is based on a gaussian approximation to optimal conditional mean estimation and follows similar developments described by Wishner et.al. [7] for extended Kalman filtering. The filtering equations are a modification of their results; the smoothing equations are new. An Euler-type integration scheme is assumed for the system dynamics.

3.2 Problem statement. Consider the problem of determining sequential estimates of the n-dimensial state vector $x_j = x_{t_j}$ of a dynamic system given corrupted discrete m-dimensional observations (or measurements)

$$z_k = h(x_k) + v_k \tag{1}$$

The $h(\cdot)$ is an m-vector nonlinear function of the state, and the set $\{v_k, k=1,\ldots\}$ is a white gaussian sequence with r X r covariance matrix

$$E\{v_k, v_j^T\} = R_k \delta_{kj} \tag{2}$$

The evolution of the state vector is described by the vector (Itô) stochastic differential equation

$$dx_t = f(x_t, t)dt + g(x_t, t)d\beta_t \tag{3}$$

where $f(\cdot,t)$ is an n-vector, $g(\cdot,t)$ is n X r, and β_t is an r-vector Brownian motion process with

$$E\{d\beta_t, d\beta_t^T\} = Q_t \, dt \tag{4}$$

The initial state is assumed to be a vector of gaussian random variables with mean $\bar{x}_o$ and covariance matrix $\mathrm{cov}(x_o, x_o | Z_o)$.

3.3 Gaussian approximation for conditional mean estimation. The criterion for optimal estimation will be the minimization of the conditional expectation of the quadratic form.

*Ref. 2, Section 4.0.

$\tilde{x}_{j|k}^T S_j \tilde{x}_{j|k}$ where S_j is an arbitrary positive definite weighting matrix. The vector $\tilde{x}_{j|k} = x_j - \hat{x}_{j|k}$ denotes the error made in estimating the state x_j with $\hat{x}_{j|k}$ an estimate based on the given realization of observations

$$Z_k = (z_1^T \; z_2^T \; \ldots\ldots \; z_k^T)^T \tag{5}$$

The estimate that minimizes this criterion is known [4] to be equal to the expectation of x_j conditioned on Z_k

$$\hat{x}_{j|k} = E\{x_j | Z_k\} \; . \tag{6}$$

This estimate is commonly described as the <u>minimum variance</u> or <u>minimum mean-square error</u> estimate.

Unfortunately $E\{x_j|Z_k\}$ cannot be determined exactly for the general non-linear problem. This is a consequence of the fact that an infinite number of terms is required to completely characterize the conditional probability density function $p(x_j|Z_k)$ [4]. If however it can be shown that the x_j and Z_k are vectors of gaussian random variables, the density function can be fully described by the following expressions for the conditional mean and auto co-variance matrix:

$$E\{x_j|Z_k\} = E\{x_j|Z_{k-1}\} - \mathrm{cov}(x_j,z_k|Z_{k-1})\mathrm{cov}(z_k,z_k|Z_{k-1})(z_k - E\{z_k|Z_{k-1}\}) \tag{7}$$

$$\mathrm{cov}(x_j,x_j|Z_k) = \mathrm{cov}(x_j,x_j|Z_{k-1}) - \mathrm{cov}(x_j,z_k|Z_{k-1})\mathrm{cov}^{-1}(z_k,z_k|Z_{k-1})\mathrm{cov}(z_k,x_j|Z_{k-1}) \tag{8}$$

where

$$\mathrm{cov}(x_\alpha,x_\beta|Z_\gamma) = E\{(x_\alpha - E\{x_\alpha|Z_\gamma\})(x_\beta - E\{x_\beta|Z_\gamma\})|Z_\gamma\} \tag{9}$$

We are motivated by the relative simplicity of equations 7 and 8 to assume that for x_o gaussian, x_j will be at least nearly gaussian. Sorenson and

Stubberud [4] explored this idea rather thoroughly by assuming the a posteriori density function to be expanded in an Edgeworth series

$$p(x_j|Z_k) = \varphi(x_j|Z_k)[1 + \frac{1}{3!} c_3 H_3(x_j|Z_k) + \frac{1}{4!} c_4 H_4(x_j|Z_k) + \dots] \tag{10}$$

where $\varphi(\cdot)$ is the conditional gaussian density function and the $H_\alpha(\cdot)$ are hermite polynomials. Their experience with filtering algorithms would indicate that the inclusion of the higher order terms in equation 10 is often only of marginal value and is computationally impractical for all but scalar and simple second-order systems. Other authors [e.g. 3,5] have investigated different expansions for $p(x_j|Z_k)$ with similar results. Since our desire is to develop a filter-smoothing algorithm that may be realistically implemented for a system relatively high order, we are by practicality constrained to the assumption that equation 10 may be truncated after the first term, i.e. that the x_j and Z_k are jointly gaussian. The experience of other authors [summarized in reference 8] with the reentry vehicle tracking problem would indicate that this is a reasonable assumption. Thus recalling equation 6 we have for the evolution of the optimal estimate due to the observation of Z_k.

$$\hat{x}_{j|k} = \hat{x}_{j|k-1} + \mathrm{cov}(x_j,z_k|Z_{k-1})\mathrm{cov}^{-1}(z_k,z_k|Z_{k-1})(z_k - \hat{z}_{k|k-1}) \tag{11}$$

and

$$\mathrm{cov}(x_j,x_j|Z_k) = \mathrm{cov}(x_j,x_j|Z_{k-1}) - \mathrm{cov}(x_j,z_k|Z_{k-1})\mathrm{cov}^{-1}(z_k,z_k|Z_{k-1})\mathrm{cov}(z_k,x_j|Z_{k-1}) \tag{12}$$

where now

$$\mathrm{cov}(x_j,x_k|Z_k) = E\{\tilde{x}_{j|k}\, \tilde{x}^T_{k|k}|Z_k\} \tag{13}$$

3.4 <u>Taylor series approximations for estimate updating</u>. Approximations to the element terms in equation 11 and 12 are obtained by expanding the various nonlinear vector functions in a Taylor series about some nominal trajectory state. A logical choice for this nominal state would be the current best estimates of x_k. Expanding $h(\cdot)$ to first order terms in $\tilde{x}_{k|k-1}$ we obtain for the measurement

$$z_k = h(\hat{x}_{k|k-1}) + H_k\, \tilde{x}_{k|k-1} + v_k \tag{14}$$

and thus taking expectations

$$\hat{z}_{k|k-1} = h(\hat{x}_{k|k-1}) \ . \tag{15}$$

H_k is defined as the m × n Jacobian matrix of $h(\cdot)$ with elements

$$[H]_{\alpha,\beta} = \left. \frac{\partial\, h_\alpha(x)}{\partial\, x_\beta} \right|_{x = \hat{x}_{k|k-1}} \tag{16}$$

The approximate measurement residual $\tilde{z}_{k|k-1}$ may now be determined by subtracting equation 16 from 14,

$$\tilde{z}_{k|k-1} = H_k\, \tilde{x}_{k|k-1} + v_k \tag{17}$$

Substituting the above expression into the defining term for the measurement conditional covariance

$$\mathrm{cov}(z_k, z_k | Z_{k-1}) = E\{\tilde{z}_{k|k-1}\, \tilde{z}^T_{k|k-1} | Z_{k-1}\} \tag{18}$$

we deduce after some manipulation that

$$\mathrm{cov}(z_k, z_k | Z_{k-1}) = H_k \mathrm{cov}(x_k, x_k | Z_{k-1}) H_k^T + R_k \tag{19}$$

The following first-order approximation to the cross-covariance matrix $\mathrm{cov}(x_j, z_k | Z_{k-1})$ is determined similarly from equation 17

$$\mathrm{cov}(x_j, z_k | Z_{k-1}) = \mathrm{cov}(x_j, x_k | Z_{k-1}) H_k^T \tag{20}$$

If in equation 20 j equals k, i.e. only the filtering problem is being considered, then equations 19 and 20 would be sufficient and the development of the estimate update equations would be complete. That is, substituting equations 15, 19, and 20 for terms in 11 and 12 one obtains for the filter

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \text{cov}(x_k, x_k | Z_{k-1}) S_k \eta_k \tag{21}$$

where we denote the <u>innovation</u>

$$\eta_k = z_k - h(\hat{x}_{k|k-1}) \tag{22}$$

and

$$\text{cov}(x_k, x_k | Z_k) = \text{cov}(x_k, x_k | Z_{k-1}) [I - S_k H_k \text{cov}(x_k, x_k | Z_{k-1})] \tag{23}$$

with

$$S_k = H_k^T (H_k \text{cov}(x_k, x_k | Z_{k-1}) H_k^T + R_k)^{-1} \tag{24}$$

It may be noted the although equivalent to Wishner's [7] equations these expressions are formulated somewhat differently. The reason will become clear when we determine the equations for smoothing.

In order to implement fixed-point smoothing an expression equivalent to equation 19 is needed for updating the cross-covariance matrix $\text{cov}(x_j, x_k | Z_k)$. Such an expression may be deduced from the following lemma:

<u>Lemma I</u>. If u_1, u_2 and u_3 are vectors of jointly gaussian random variables, then

$$\text{cov}(u_1, u_2 | u_3) = \text{cov}(u_1, u_2) - \text{cov}(u_1, u_3) \text{cov}^{-1}(u_3, u_3) \text{cov}(u_3, u_2) \tag{25}$$

A proof of Lemma I is included in Section 3 below.

By identifying x_j, x_k, and z_k with u_1, u_2, and u_3 respectively in equation 25 and recalling that $Z_k = (Z_{k-1}^T, z_k^T)^T$, one sees that

$$\text{cov}(x_j, x_k | Z_k) = \text{cov}(x_j, x_k | Z_{k-1}) - \text{cov}(x_j, z_k | Z_{k-1}) \text{cov}^{-1}(z_k, z_k | Z_{k-1}) \text{cov}(z_k, x_k | Z_{k-1}) \tag{26}$$

Taken together equations 19, 20, and 26 describe the necessary relationships for updating the cross-covariance terms $cov(x_j,x_k|Z_k)$ and $cov(x_k,x_j|Z_k)(=cov(x_j,x_k|Z_k)^T)$. Making the appropriate substitutions we determine

$$cov(x_j,x_k|Z_k) = cov(x_j,x_k|Z_{k-1})[I - S_kH_k cov(x_k,x_k|Z_{k-1})] \qquad (27)$$

We shall use equation 27 along with the expression

$$x_{j|k} = \hat{x}_{j|k-1} + cov(x_j,x_k|Z_{k-1})S_k\eta_k \qquad (28)$$

to define the nonlinear fixed-point smoothing estimate updating. The smoothing is initiated by the filtered estimates $x_{j|j}$ and $cov(x_j,x_j|Z_j)$, or if $j = 0$, then obviously by $\bar{x}_0$ and $cov(x_0,x_0)$. It should be noted that with the above formulation, fixed-point smoothing introduces no additional matrix inversion operations to the estimation algorithm.

Assuming the necessary extrapolation equations describing the evolution of the estimates between samples are available, the algorithm is complete even though no expression has been determined for the auto-covariance $cov(x_j,x_j|Z_k)$. We include such an expression for completeness;

$$cov(x_j,x_j|Z_k) = cov(x_j,x_j|Z_{k-1}) - cov(x_j,x_k|Z_{k-1})S_kH_k cov^T(x_j,x_k|Z_{k-1}) \qquad (29)$$

The development of the dynamic equations for the various estimates and associated covariance matrices between samples will now be considered.

3.5 Estimate extrapolation. Recall that the evolution of the system state vector is assumed to be described by the expression

$$dx_t = f(x_t,t)dt + g(x_t,t)d\beta_t \qquad (3)$$

Proceeding as before, we expand $f(x_t,t)$(for $t_{k-1} \leq t \leq t_k$) in a first-order Taylor series about the estimate $\hat{x}_{t|k-1}$;

$$dx_t = [f(\hat{x}_{t|k-1},t) + F_t\,\tilde{x}_{t|k-1}]dt + g(x_t,t)d\beta_t \quad t_{k-1} \leq t \leq t_k \qquad (30)$$

where

$$[F_t]_{\alpha,\beta} = \left.\frac{\partial f_\alpha(x_t,t)}{\partial x_\beta}\right|_{x_t = \hat{x}_{t|k-1}} \tag{31}$$

In light of our discussion on minimum variance estimators and equation 6, we determine the following dynamic equations for the suboptimal estimate

$$d\,\hat{x}_{t|k-1} = f(\hat{x}_{t|k-1},t)dt \tag{32}$$

and estimate error between measurement samples

$$d\,\tilde{x}_{t|k-1} = F_t\,\tilde{x}_{t|k-1}\,dt + g(x_t,t)d\beta_t \tag{33}$$

In order to facilitate on-line estimation, the dynamic equations will be discretized by simple Euler numerical integration. Integrating equations 32 and 33, we have for $t_k = t_{k-1} + \Delta t$

$$\hat{x}_{k|k-1} = \hat{x}_{k-1|k-1} + \Delta t\; f(\hat{x}_{k-1|k-1},t_{k-1}) \tag{34}$$

and

$$\tilde{x}_{k|k-1} = [I + F_{k-1}\Delta t]\tilde{x}_{k-1|k-1} + g(x_{k-1},t_{k-1})w_{k-1} \tag{35}$$

where the w_k are defined to be elements of a white gaussian sequence with

$$E\{w_j\,w_k^T\} = \Delta t\; Q_{j-1}\,\delta_{jk}{}^{*}. \tag{36}$$

In order to simplify the notation, we shall define the following terms

---

*Although treating the noise process in this manner is certainly an abuse of the theory of stochastic differential equations (see ref [4], chapter 4 for example), it will suffice in our context of the estimation problem.

$$\Phi_{k-1} = I + \Delta t\, F_{k-1} \tag{37}$$

$$Q'_{k-1} = \Delta t\, Q_{t\ k-1} \tag{38}$$

Substituting equation 37 into 35 we have

$$\tilde{x}_{k|k-1} = \Phi_{k-1}\, \tilde{x}_{k-1|k-1} + g(x_{k-1}, t_{k-1}) w_{k-1} \tag{39}$$

Errors introduced into the estimate extrapolation by discretizing equation 33 may be reduced somewhat by including second order terms (in Δt) in the expansion, i.e.

$$\hat{x}_{k|k-1} = \hat{x}_{k-1|k-1} + \Delta t\, f(\hat{x}_{k-1|k-1}, t_{k-1}) + \frac{\Delta t^2}{2} F_{k-1} f(\hat{x}_{k-1|k-1}, t_{k-1}) \tag{40}$$

Since the inclusion of the additional factor does not require that we determine any additional matrix terms [F_{k-1} must be computed anyway], increased accuracy may be obtained at little computational cost. As a consequence, equation 40 will be employed in preference to equation 34 for estimate extrapolation.

Approximate equations for $\mathrm{cov}(x_k, x_k | Z_{k-1})$ and $\mathrm{cov}(x_j, x_k | Z_{k-1})$ may be determined by substituting equation 39 into the appropriate defining expression. Thus for

$$\mathrm{cov}(x_k, x_k | Z_{k-1}) = E\{\tilde{x}_{k|k-1}\, \tilde{x}^T_{k|k-1} | Z_{k-1}\} \tag{41}$$

we have

$$\begin{aligned} \mathrm{cov}(x_k, x_k | Z_{k-1}) = E\{[\,&\Phi_{k-1}\, \tilde{x}_{k|k-1} + g(x_{k-1}, t_{k-1}) w_{k-1}] \\ &[\,\Phi_{k-1}\, \tilde{x}_{k|k-1} + g(x_{k-1}, t_{k-1}) w_{k-1}]^T | Z_{k-1}\}^T \end{aligned} \tag{42}$$

which reduces to

$$\mathrm{cov}(x_k,x_k|Z_{k-1}) = \Phi_{k-1}\,\mathrm{cov}(x_{k-1},x_{k-1}|Z_{k-1})\Phi^T_{k-1} + E\{g(x_{k-1},t_{k-1})Q'_{k-1}\,g^T(x_{k-1},t_{k-1})|Z_{k-1}\} \tag{43}$$

Further reductions are not possible without expanding $g(x_{k-1},t_{k-1})Q'_{k-1}\,g^T(x_{k-1},t_{k-1})$ about the $\hat{x}_{k-1|k-1}$ nominal trajectory. Proceeding with the Taylor series expansion, we determine to first order in $\tilde{x}_{k-1|k-1}$ that

$$\mathrm{cov}(x_k,x_k|Z_{k-1}) = \Phi_{k-1}\mathrm{cov}(x_{k-1},x_{k-1}|Z_{k-1})\Phi^T_{k-1} + g(\hat{x}_{k-1|k-1},t_{k-1})Q'_{k-1}\,g^T(\hat{x}_{k-1|k-1},t_{k-1}) \tag{44}$$

The expression for the cross covariance term is determined in a similar manner

$$\mathrm{cov}(x_j,x_k|Z_{k-1}) = E\{\tilde{x}_{j|k-1}[\Phi_{k|k-1}\,\tilde{x}_{k-1|k-1} + g(x_{k-1},t_{k-1})w_k]^T\} \tag{45}$$

$$= \mathrm{cov}(x_j,x_{k-1}|Z_{k-1})\Phi^T_{k|k-1} \tag{46}$$

We summarize our development by collecting the various results into the following theorem.

<u>Theorem I</u>. The first-order filter-smoother for the discretized nonlinear system described herein consists of the following extrapolation equations between samples

$$\hat{x}_{k|k-1} = \hat{x}_{k-1|k-1} + \Delta t\ f(\hat{x}_{k-1|k-1},t_{k-1}) + \frac{\Delta t^2}{2}F_{k-1}\,f(\hat{x}_{k-1|k-1},t_{k-1}) \tag{40}$$

$$\mathrm{cov}(x_k,x_k|Z_{k-1}) = \Phi_{k-1}\mathrm{cov}(x_{k-1},x_{k-1}|Z_{k-1})\Phi^T_{k-1} + g(\hat{x}_{k-1|k-1},t_{k-1})Q'_{k-1}\,g^T(\hat{x}_{k-1|k-1},t_{k-1}) \tag{44}$$

$$\mathrm{cov}(x_j,x_k|Z_{k-1}) = \mathrm{cov}(x_j,x_{k-1}|Z_{k-1})\Phi^T_{k-1} \tag{46}$$

and the following update equations at the samples

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \mathrm{cov}(x_k, x_k | Z_{k-1}) S_k \eta_k \tag{21}$$

$$\hat{x}_{j|k} = \hat{x}_{j|k-1} + \mathrm{cov}(x_j, x_k | Z_{k-1}) S_k \eta_k \tag{28}$$

$$\mathrm{cov}(x_k, x_k | Z_k) = \mathrm{cov}(x_k, x_k | Z_{k-1})[I - S_k H_k \mathrm{cov}(x_k, x_k | Z_{k-1})] \tag{23}$$

$$\mathrm{cov}(x_j, x_k | Z_k) = \mathrm{cov}(x_j, x_k | Z_{k-1})[I - S_k H_k \mathrm{cov}(x_k, x_k | Z_{k-1})] \tag{27}$$

where

$$\eta_k = z_k - h(\hat{x}_{k|k-1}) \tag{22}$$

and

$$S_k = H_k^T (H_k \mathrm{cov}(x_k, x_k | Z_{k-1}) H_k^T + R_k)^{-1} \tag{24}$$

3.6 Proof of Lemma I

Define the augmented vector

$$U = \begin{pmatrix} u_1 \\ \hdashline u_2 \end{pmatrix} \tag{47}$$

and substitute it into the following augmented covariance matrices

$$\mathrm{cov}(U,U) = E\left\{\begin{pmatrix} u_1 \\ \hdashline u_2 \end{pmatrix} (u_1^T \,\vdots\, u_2^T)\right\} = \left[\begin{array}{c:c} \mathrm{cov}(u_1,u_1) & \mathrm{cov}(u_1,u_2) \\ \hdashline \mathrm{cov}(u_2,u_1) & \mathrm{cov}(u_2,u_2) \end{array}\right] \tag{48}$$

$$\mathrm{cov}(U,u_3) = E\left\{\begin{pmatrix} u_1 \\ \hdashline u_2 \end{pmatrix} u_3^T\right\} = \left[\begin{array}{c} \mathrm{cov}(u_1,u_3) \\ \mathrm{cov}(u_2,u_3) \end{array}\right] \tag{49}$$

$$\mathrm{cov}(u_3,U) = E\{u_3(u_1^T \,\vdots\, u_2^T)\} = [\mathrm{cov}(u_3,u_1) \,\vdots\, \mathrm{cov}(u_3,u_2)] \tag{50}$$

These expressions may now be substituted into the equation for the conditional covariance $\mathrm{cov}(U,U|u_3)$,

$$\mathrm{cov}(U,U|u_3) = \mathrm{cov}(U,U) - \mathrm{cov}(U,u_3)\mathrm{cov}^{-1}(u_3,u_3)\mathrm{cov}(u_3,U) \tag{51}$$

Performing the indicated matrix operations we obtain

$$\left[\begin{array}{c:c} \mathrm{cov}(u_1,u_1|u_3) & \mathrm{cov}(u_1,u_2|u_3) \\ \hdashline \mathrm{cov}(u_2,u_1|u_3) & \mathrm{cov}(u_2,u_2|u_3) \end{array}\right] = \left[\begin{array}{c:c} \mathrm{cov}(u_1,u_1) - \mathrm{cov}(u_1,u_3)\mathrm{cov}^{-1}(u_3,u_3)\mathrm{cov}(u_3,u_1) & \mathrm{cov}(u_1,u_2) - \mathrm{cov}(u_1,u_3)\mathrm{cov}^{-1}(u_3,u_3)\mathrm{cov}(u_3,u_2) \\ \hdashline \mathrm{cov}(u_2,u_1) - \mathrm{cov}(u_2,u_3)\mathrm{cov}^{-1}(u_3,u_3)\mathrm{cov}(u_3,u_1) & \mathrm{cov}(u_2,u_2) - \mathrm{cov}(u_2,u_3)\mathrm{cov}^{-1}(u_3,u_3)\mathrm{cov}(u_3,u_2) \end{array}\right] \tag{52}$$

The off-diagonal term is recognized as the desired result.

4.0 DETERMINISTIC COMPUTATIONS FOR EVALUATING FILTER-SMOOTHER PERFORMANCE

4.1 Purpose. The conventional way to evaluate a RATRAN signal-processing filter is by a program of Monte Carlo computation, as described in the next paragraph. This memo outlines a deterministic method for the same purpose. The deterministic method requires more programming. It gives in return oetter insight into the meaning of the results and freedom from sampling variances.

These results are directly applicable to evaluation of recursive filter-smoother algorithms such as the one developed in Section 3 above. While expressions are included in the above work for estimate covariances, such expressions are limited to the specific conditions required for the optimalization. The formalization presented here enables computing the effects of radar bias errors, and also errors due to other system and environmental deviations from conditions assumed in the optimalization.

4.2 Conventional method. In the conventional method, for which the programming is in progress, a trajectory is selected and a number of state vectors

$$x_k = x(t_k) \quad k = 1,\ldots,n \tag{53}$$

are computed. The noisefree radar observation vectors

$$r_k = h(x_k) \quad k = 1,\ldots,n \tag{54}$$

are also computed. These are used repeatedly with different samples of radar noise vectors v_k, and possibly a radar bias vector $b(x_k)$, giving

$$z_k = h(x_k) + b(x_k) + v_k \tag{55}$$

for the radar observation vectors. The z's are used as inputs to the filter equations, which compute the estimated state vectors $\hat{x}_k$ and the errors in the estimated state vectors

$$\delta x_k = \hat{x}_k - x_k . \tag{56}$$

For a number of samples of radar noise, the mean of the observed errors in (56) is taken to be the bias $\hat{E}(\delta x)$ of the filter in the estimate of $x(t_k)$. The covariance of the error is taken to be the mean of $(\delta x - \hat{E}(\delta x)(\delta x - \hat{E}(\delta x))^T$, for the noise samples used.

4.3 Outline of deterministic method. The deterministic method, outlined next, does not use radar noise samples. Instead it uses the mean and the covariance of the radar noise. The deterministic computations follow the propagations of the means and covariances through the filter, instead of the actual errors as in the Monte Carlo computations.

Let

$$x_k = f(x_{k-1}, w_{k-1}) \tag{57}$$

be the true transition function, where as before x_k is the state vector at time t_k and now w_k is a parameter vector. The components of w_k are those scalar parameters whose effects on the trajectory are of interest: wind velocity components, air density, etc. (In the conventional or Monte Carlo computations, it is convenient to separate the bias errors from the random errors; in the deterministic method it is convenient to discuss them, and even to compute their effects, concurrently.)

The filter, in its prediction step, computes, using an estimated parameter vector $\hat{w}$,

$$\hat{x}_{k|k-1} = g(\hat{x}_{k-1}, \hat{w}_{k-1}) \tag{58}$$

where $g(\cdot,\cdot)$ differs from $f(\cdot,\cdot)$ in RATRAN because of the use of the simplified-dynamics equations in the filter, and also because of truncation errors in the filter computations. Let

$$\varphi(x,w) = g(x,w) - f(x,w) \tag{59}$$

by definition, for any (x,w).

The filter then computes

$$\hat{x}_k = \hat{x}_{k|k-1} + A_k(z_k - h(\hat{x}_{k|k-1})) \tag{60}$$

where A_k is the Kalman gain matrix, here assumed to be precomputed -- i.e., to be a function of x_k rather than of $\hat{x}_k$.

Let H_k be the matrix with element $\partial h_i(x_k)/\partial x_{(j)}$ in row i and column j, where $h_i(\cdot)$ is the ith element of the radar coordinate vector r_k and $x_{(j)}$ is the jth element of the state vector $x(t_k)$, so that

$$h(\hat{x}_{k|k-1}) \cong h(x_k) + H_k\, \delta x_{k|k-1} \tag{61}$$

where

$$\delta x_{k|k-1} = \hat{x}_{k|k-1} - x_k \tag{62}$$

is the error in the prediction step of the filter. Then (55), (60), and (61) give, approximately,

$$\delta x_k = A_k(b(x_k) + v_k) + (I - A_k H_k)\delta x_{k|k-1} \tag{63}$$

The computation (58) may be written

$$\delta x_{k|k-1} = g(\hat{x}_{k-1},\hat{w}_{k-1}) - f(x_{k-1},w_{k-1})$$

$$= g(\hat{x}_{k-1},\hat{w}_{k-1}) - g(x_{k-1},w_{k-1}) + \varphi(x_{k-1},w_{k-1})$$

Then

$$\delta x_{k|k-1} = \varphi_{k-1} + G^{(x)}_{k-1}\,\delta x_{k-1} + G^{(w)}_{k-1}\,\delta w_{k-1} \tag{64}$$

where $G^{(x)}_{k-1}$ is the matrix that has in row i, column j the partial derivative of the ith component of $g(x,w)$ with respect to the jth component of x, evaluated at t_{k-1}; $G^{(w)}_{k-1}$ is similarly defined except that the partial derivatives are with respect to the components of w; and

$$\varphi_{k-1} = \varphi(x_{k-1},w_{k-1})$$

$$\delta w_{k-1} = \hat{w}_{k-1} - w_{k-1}.$$

Substitution of (64) into (63) gives

$$\delta x_k = A_k(b(x_k) + v_k) + (I - A_k H_k)(\varphi_{k-1} + G^{(w)}_{k-1}\,\delta w_{k-1})$$

$$+ (I - A_k H_k)G^{(x)}_{k-1}\,\delta x_{k-1} \tag{65}$$

Equation (65) hold equally well for the Monte Carlo computations as for what follows below. In the deterministic computations, equation (65) is not used, but instead the equation that follows from taking the expected value of both sides is used:

$$E(\delta x_k) = A_k b(x_k) + (I - A_k H_k)(\varphi_{k-1} + G^{(w)}_{k-1}\ \delta w_{k-1})$$

$$+ (I - A_k H_k) G^{(x)}_{k-1}\ E(\delta x_{k-1}) \qquad (66)$$

In (66), it has been assumed that $\hat{w}$ is deterministic and that $Ev_k = 0$. It would be easy, but I believe not useful in RATRAN, to let $\hat{w}$ have random components. Each symbol in (66) except $E(\delta x_k)$ and $E(\delta x_{k-1})$ represents a quantity that is known before the error computations begin. Therefore (66) may be used, given an input value for the mean initial error vector $E(\delta x_o)$, to compute the mean error in each $\hat{x}_k$.

If we write

$$y_k = A_k(b(x_k) + v_k) + (I - A_k H_k)(\varphi_{k-1} + G^{(w)}_{k-1}\ \delta w_{k-1}) \qquad (67)$$

for the forcing term in (65) and

$$C_k = (I - A_k H_k) G^{(x)}_{k-1} \qquad (68)$$

for the loop gain, so that (65) becomes

$$\delta x_k = y_k + C_k\ \delta x_{k-1} \qquad (69)$$

it follows that

$$\mathrm{cov}(\delta x_k, \delta x_k) = \mathrm{cov}(y_k, y_k) + C_k\ \mathrm{cov}(\delta x_{k-1}, \delta x_{k-1}) C_k^T$$

$$+ \mathrm{cov}(y_k, \delta x_{k-1}) C_k^T + C_k\ \mathrm{cov}(\delta x_{k-1}, y_k) \qquad (70)$$

If all the terms in (67) are deterministic except for v_k, and if $Ev_k = 0$,

$$\mathrm{cov}(y_k, y_k) = A_k \,\mathrm{cov}(v_k, v_k) A_k^T \tag{71}$$

and if further the v_k's are uncorrelated for different values of k, it follows that

$$\mathrm{cov}(y_k, \delta x_{k-1}) = \mathrm{cov}(\delta x_{k-1}, y_k) = 0$$

Then (70) becomes

$$\mathrm{cov}(\delta x_k, \delta x_k) = A_k \,\mathrm{cov}(v_k, v_k) A_k^T + C_k \,\mathrm{cov}(\delta x_{k-1}, \delta x_{k-1}) C_k^T \tag{72}$$

Equations (66) and (72) give the desired results.

Equation (66) gives also the effects of radar bias and other bias errors; small bias errors of course have no effect on covariances.

For smoothing instead of filtering, an equation similar to (60), but using a different matrix in place of A_k, leads through identical steps to equations similar to (66) and (72).

4.4 Explicit role of input errors. Equations (66) and (72), being recursive, do not reveal to casual inspection how each input error affects the state vector estimate. For additional insight, and under some conditions for an economy in computation at the cost of some programing effort and some storage space, there may be value in using instead (73) and (75) below.

Equation (66) implies

$$E(\delta x_k) = \bar{y}_k + C_k \bar{y}_{k-1} + C_k C_{k-1} \bar{y}_{k-2} + \ldots + C_k C_{k-1} \cdots C_2 \bar{y}_1$$
$$+ C_k C_{k-1} \cdots C_2 C_1 E(\delta x_o) \tag{73}$$

where

$$\bar{y}_k = A_k b(x_k) + (I - A_k H_k)(\varphi_{k-1} + F_{k-1}^{(w)} \delta w_{k-1}) \tag{74}$$

Equation (73) shows the contribution of the initial error input to the Kalman filter, and the effects of the contributions from each radar look.

Similarly equation (72) implies

$$\mathrm{cov}(\delta x_k,\delta x_k) = R_k' + C_k R_{k-1}' C_k^T + \ldots + C_k C_{k-1} \ldots C_2 R_1' C_2^T \ldots C_k^T$$

$$+ C_k C_{k-1} \ldots C_1 \mathrm{cov}(\delta x_o,\delta x_o) C_1^T \ldots C_k^T \tag{75}$$

where

$$R_j' = A_j \ \mathrm{cov}(v_j,v_j) A_j^T \quad , \ j=1,\ldots,k. \tag{76}$$

4.5 Effects on launch-point estimation. Let x_i be the state vector just after the launch. If x_i is perturbed by a small vector δx_i, the effect on $x_k = x(t_k)$ is a change

$$\delta x_k = U_k^i \ \delta x_i \tag{77}$$

where U_k^i is a square matrix of partial derivatives, with the partial derivative of the rth component of x_k with respect to the sth component of x_i in row r, column s. Then

$$E(\delta x_k) = U_k^i \ E(\delta x_i) \tag{78}$$

and

$$\mathrm{cov}(\delta x_k,\delta x_k) = U_k^i \ \mathrm{cov}(\delta x_i,\delta x_i)(U_k^i)^T \tag{79}$$

The elements of U_k^i are being computed and stored, for a number of trajectories and a number of values of k, for another purpose. From these elements there will be computed the inverse matrices

$$U_i^k = (U_k^i)^I \ ,$$

for use in

$$E(\delta x_i) = U_i^k \ E(\delta x_k) \tag{80}$$

and

$$\mathrm{cov}(\delta x_i, \delta x_i) = U_i^k \, \mathrm{cov}(\delta x_k, \delta x_k)(U_i^k)^T \tag{81}$$

Finally, the mean error in estimated launch point, and its covariance, will be computed from

$$E \begin{pmatrix} \delta x_L \\ \delta z_L \end{pmatrix} = LE(\delta x_i) \tag{82}$$

and

$$\mathrm{cov} \left\{ \begin{pmatrix} \delta x_L \\ \delta z_L \end{pmatrix}, \begin{pmatrix} \delta x_L \\ \delta z_L \end{pmatrix} \right\} = L \, \mathrm{cov}(\delta x_i, \delta x_i) L^T \tag{83}$$

where

$$L = \begin{pmatrix} 1 & -\dot{x}_o/\dot{y}_o & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\dot{z}_o/\dot{y}_o & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{84}$$

adjusts for the time when the estimated altitude is zero, instead of t_i when the estimated time-of-flight is zero. In (84), $\dot{x}_o, \dot{y}_o$, and $\dot{z}_o$ are the downrange, upward, and lateral components of launch velocity.

5.0 CONCLUSIONS

The results of Section 2 show the performance of the simplified dynamics approximations to a variety of trajectories computed by the BRL point-mass model. In general it is seen that the approximations provide reasonably accurate computation of launch parameters provided that the time from which back-tracking is initiated is sufficiently early. Error variations are due chiefly to differences between the projectiles and to geometric dependence on quadrant elevation angle.

Section 3 demonstrates the mathematical validity of the optimal smoothing-filtering algorithm presented in the 2nd Quarterly Progress Report.

Section 4 presents a method for deterministic evaluation of recursive filter-smoother algorithms, accounting for bias and modelling errors as well as random noise-like observation errors. This procedure can avoid the need for extensive simulation runs for filter-smoother evaluation.

The remaining work planned in the next few months of this effort will be mainly concerned with the application of computer programs and techniques described in this and previous Quarterly Progress Reports. Specific results will include:

(a) Minimum variances of trajectory parameter estimation errors caused by errors in radar observations (assuming true optimum processing). Sensitivity of results to radar-target geometry and radar system parameters such as track time, data rate, and S/N will be indicated as well as an assessment of the utility of doppler measurement for range-rate.

(b) Simulations of optimal filter-smoother algorithm performance for a variety of radar-target geometries. Evaluations of performance will be presented. Comparisons will be made with true optimum (see (a) above) and sensitivities will be determined with respect to bias and modelling errors. While some Monte-Carlo simulation runs will be performed, most of the evaluations will be based on the deterministic evaluation technique described in Section 4 above.

(c) Complete descriptions and listings of the computer algorithms and programs plus a guide to their use and an evaluation of their possible utility in an operational system.